Numerical analysis of phase transitions in a stochastic pension fund with mean field system

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1 Introduction

The situation of pension funds in Brazil has deteriorated in recent years. According to ABRAPP, brazilian association of pension funds, see ABRAPP [2016] the number of deficit pension plans increased from 121 in 2010 to 239 in 2015, with a total deficit going from R$6.3 billion to R$76.7 billion.The low return obtained by these pension plans is responsible for part of this deficit.

Historically the return of the public brazilian bonds are very high, but in the recent years they are in the lowest level of the history. This make it increasingly difficult to reach the actuarial targets. This kind of market movement, the change in the level of the returns, it is not new but normally it usually not to be taken into account by pension funds.

We are interested in studying the situation where the fall of one or a small group of assets leads to the fall of the whole portfolio and the impacts of this fall in the pension plans. The main objective of modelling the portfolio behaviour of pension funds is to use an appropriate model for the return rate. The classical model that describe this phenomena, so called "systemic risk of portfolio" on the area of mathematical finance was introduced by GARNER. Many other theoretical results and methodological discussion are developed around the context of such stochastic model (see GARNER for more details).

In this work we consider such system in order to study via simulation the phenomena of a "phase transition" of two rate of return of the equilibrium distribution caused by a complex dynamics. We use a simple Euler scheme in order to simulate the system and succeeded in reproduce the behaviour of the dynamics that evolves from one equilibrium solution of the system to the others solutions. We show that such model is capable to reproduce dynamically the effect to pass from one solution (negative or zero) to a positive return rate. Modelling this phenomena and perform the account of the reserves of the portfolio during transition is of the most importance to characterize pension found systemic risk. We describe this phenomena in Sections 2 and 3. We also obtain numerically via simulations the equilibrium solution of the system described in GARNER, see Proposition 1 below.

2 Model

We considered a portfolio where each asset price, S_t^k , are modeled as continuous-time stochastic processes satisfying the system of Itô stochastic differential equation:

$$dS_t^k = -hU(S_t^k)dt + \theta(\bar{S}(t) - S_t^k)dt + \sigma dW_t^k,$$

where, $\{W_t^k\}_{k=1}^K$ are independent standard Brownian motions and σ is the strength of the destabilizing random forces.

The drift of the asset S_k interact through mean reversion term with the other assets,

$$\theta(\bar{S}(t) - S_t^k)$$

where interaction has intensity θ and $\bar{S}_t := \frac{1}{K}\sum_{k=1}^K S_t^k$. The $-hU(y) = -hV'(y)$ is the restoring force, V is a potential with two stable states

$$V(y) = \frac{1}{4}y^4 - \frac{1}{2}y^2 + c.$$

In order to demonstrate the behaviour of the systems empirical mean, we simulate it using Euler scheme, see Peter Eris Kloeden [2003], in the statistical software R:

$$S_{n+1}^k = S_n^k - hU(S_n^k)\Delta t + \sigma\Delta W_n^k + \theta(\bar{S}_n - S_n^k)\Delta t.$$

We take $S_0^k = -1, U(y) = y^3 - y, \Delta t = 0,02$, and let $\{\Delta W_j^k\}_{j,k}$ be independent Gaussian random variables with mean zero an variance Δt.

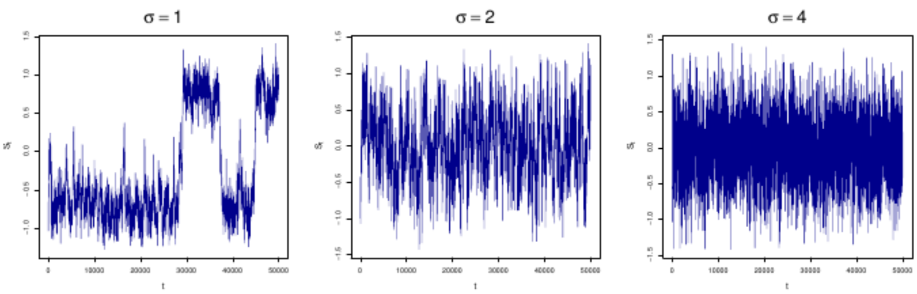

Figure 1: Simulations with different values of σ and with $K = 100, h = 0.1$ and $\theta = 6$

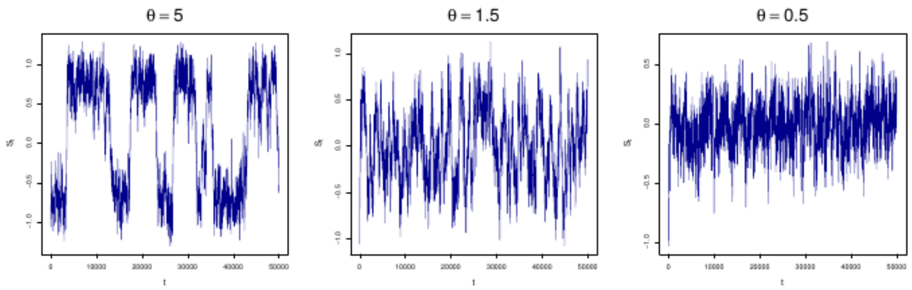

Figure 2: Simulations with different values of σ and with $K = 100, h = 0.1$ and $\theta = 1$

It is possible see in figure 1 and figure 2 the behaviour of the empirical mean as the system transitions from having two equilibria to having a single one.

Figure 3 shows how increasing h increase the system stability, that is, increase the resistance to the transition of the empirical mean of the system form one state to the other. Figure 4 illustrates that a larger system is more stable.

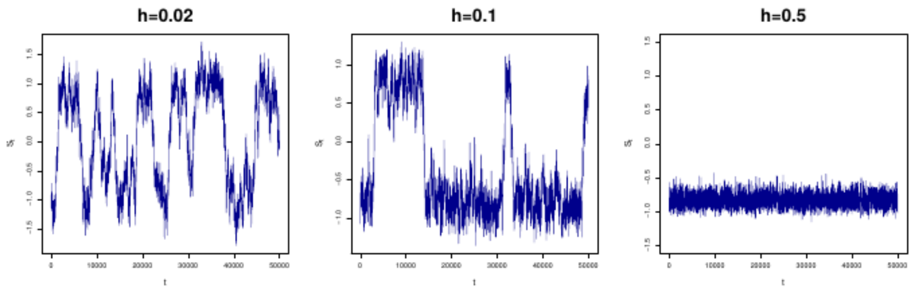

Figure 3: Simulations with different values of h and with $K = 100, \sigma = 1$ and $\theta = 10$

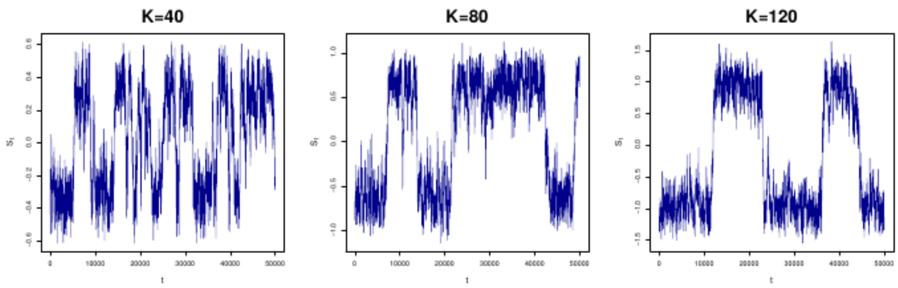

Figure 4: Simulations with different values of K and with $H = 0.1, \sigma = 1$ and $\theta = 10$

3 Equilibrium Solutions

In Garnier et al. [2012] andDawson [1983], it is shown that an equilibrium solution u^e_ξ , assuming that $\xi = \lim_{t\to\infty} \int yu(t,y)dy$, has the form:

$$u^e_\xi = \frac{1}{Z_\xi\sqrt{2\pi\frac{\sigma^2}{2\theta}}} \exp\left[-\frac{(y-\xi)^2}{2\frac{\sigma^2}{2\theta}} - h\frac{2}{\sigma^2 V(y)}\right] \tag{1}$$

with Z_ξ the normalization constant:

$$Z_\xi = \int \frac{1}{\sqrt{2\pi\frac{\sigma^2}{2\theta}}} \exp\left[-\frac{(y-\xi)^2}{2\frac{\sigma^2}{2\theta}} - h\frac{2}{\sigma^2 V(y)}\right] dy$$

For $U(y) = y^3 - y$ the consistency condition:

$$Z_\xi = m(\xi) := \int yu^e_\xi(y)dy$$

is satisfying when $\xi = 0$ and there are two additional non-zero solutions $\pm\xi_b$ if and only if $m(\theta) > 1$, and for given h and θ, there exists a critical $\sigma_c(h,\theta) > 0$ such that $\frac{d}{d\xi}m(0) > 1$ if and only if $\sigma < \sigma_{c(h,\theta)}$.

Proposition 1. *For small h, the critical value σ_c can be expanded as:*

$$\sigma_c = \sqrt{\frac{2\theta}{3}} + O(h).$$

In addition, the non-zero solutions $\pm\xi_b$ are:

$$\pm\xi_b = \pm\sqrt{1 - 3\frac{\sigma^2}{2\theta}} \left(1 + h\frac{6}{\sigma^2}\left(\frac{\sigma^2}{2\theta}\right)^2 \frac{1-2(\frac{\sigma^2}{2\theta})}{1-3(\frac{\sigma^2}{2\theta})}\right) + O(h^2)$$

Using the similarity of the exact density of the equilibrium solution (1) of the system with the normal density we implement the Metropolis Hastings algorithm, i.e., using the normal density as proposal against to the target density (1). Figure 5 shows the realizations of u_{-1}, u_0, u_1, for $h = 0.01, \theta = 6, \sigma = 1$ and $\xi \in \{-1, 0, 1\}$.

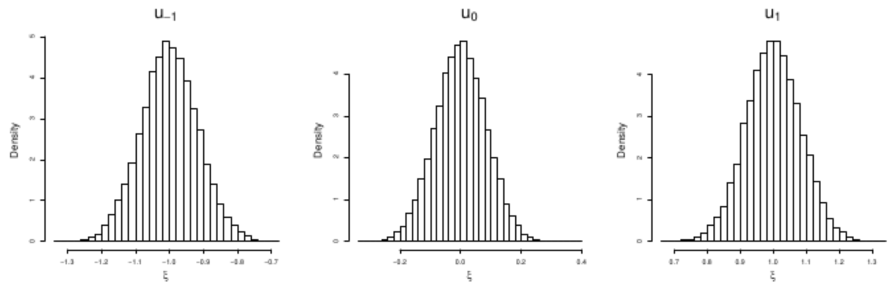

Figure 5: Simulations for ξ values with $h = 0.01, \sigma = 1$ and $\theta = 6$

Also, using the empirical density is possible to represent the phase diagram of the system, as shows Figure 6 for $\sigma = 2 > \sigma_c$, $\theta = 1, h = 10$, we obtained that $\xi \neq m(\xi)\ \forall \xi$. Now notice that for $\sigma = 0.5 < \sigma_c, \theta = 1$ and $h = 10$, we have $\xi = m(\xi)$ for $\xi = -1, 0, 1$.

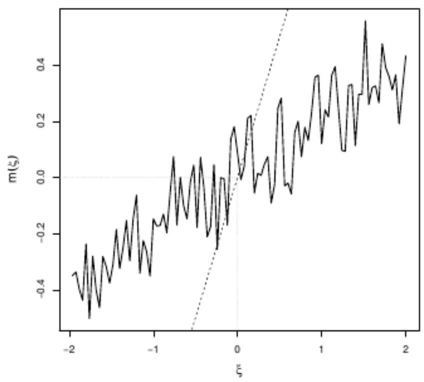

Figure 6: Asymptotic distribution mean $m(\xi)$, $h = 10$ and $\sigma = 2 > \sigma_c$

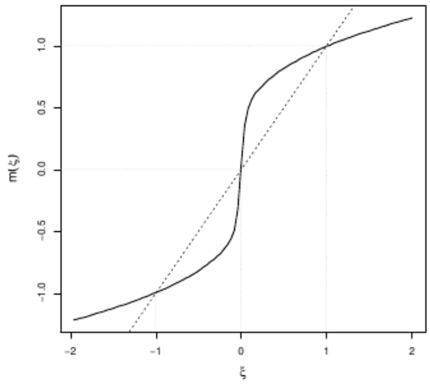

Figure 7: Asymptotic distribution mean $m(\xi)$, $h = 10$ and $\sigma = 0.5 < \sigma_c$

4 Conclusion

We use a stochastic differential equation system, with a potential, a mean field and i.i.d. volatility. We demonstrate, using simulations, how the behaviour of the empirical mean of the system is affected by $\sigma_{c(\theta,h)}$ that determines if the system is stable or bistable. We also exemplify, with simulations, the behavior of the system for different values of K, h and θ, and we conclude that the higher the h or K smaller the number of phase transitions.

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